

NONINCREASING DEPTH FUNCTIONS OF MONOMIAL IDEALS

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ABSTRACT. Given a nonincreasing function $f : \mathbb{Z}_{\geq 0} \setminus \{0\} \rightarrow \mathbb{Z}_{\geq 0}$ such that (i) $f(k) - f(k+1) \leq 1$ for all $k \geq 1$ and (ii) if $a = f(1)$ and $b = \lim_{k \rightarrow \infty} f(k)$, then $|f^{-1}(a)| \leq |f^{-1}(a-1)| \leq \dots \leq |f^{-1}(b+1)|$, a system of generators of a monomial ideal $I \subset K[x_1, \dots, x_n]$ for which $\text{depth } S/I^k = f(k)$ for all $k \geq 1$ is explicitly described. Furthermore, we give a characterization of triplets of integers (n, d, r) with $n > 0$, $d \geq 0$ and $r > 0$ with the properties that there exists a monomial ideal $I \subset S = K[x_1, \dots, x_n]$ for which $\lim_{k \rightarrow \infty} \text{depth } S/I^k = d$ and $\text{dstab}(I) = r$, where $\text{dstab}(I)$ is the smallest integer $k_0 \geq 1$ with $\text{depth } S/I^{k_0} = \text{depth } S/I^{k_0+1} = \text{depth } S/I^{k_0+2} = \dots$.

INTRODUCTION

The study on depth of powers of ideals, which originated in [3], has been achieved by many authors in the last decade. Let $S = K[x_1, \dots, x_n]$ denote the polynomial ring in n variables over a field K and $I \subset S$ a homogeneous ideal. The numerical function $f : \mathbb{Z}_{\geq 0} \setminus \{0\} \rightarrow \mathbb{Z}_{\geq 0}$ defined by $f(k) = \text{depth } S/I^k$ is called the *depth function* of I . It is known [1] that $f(k) = \text{depth } S/I^k$ is constant for $k \gg 0$. We call $\lim_{k \rightarrow \infty} f(k)$ the *limit depth* of I . The smallest integer $k_0 \geq 1$ for which $f(k_0) = f(k_0+1) = f(k_0+2) = \dots$ is said to be the *depth stability number* of I and is denoted by $\text{dstab}(I)$.

An exciting conjecture ([3, p. 549]) is that *any* convergent function $f : \mathbb{Z}_{\geq 0} \setminus \{0\} \rightarrow \mathbb{Z}_{\geq 0}$ can be the depth function of a homogeneous ideal. In [3, Theorem 4.1], given a bounded nondecreasing function $f : \mathbb{Z}_{\geq 0} \setminus \{0\} \rightarrow \mathbb{Z}_{\geq 0}$, a system of generators of a monomial ideal I for which $\text{depth } S/I^k = f(k)$ for all $k \geq 1$ is explicitly described. In [2, Theorem 4.9], it is shown that, given a nonincreasing function $f : \mathbb{Z}_{\geq 0} \setminus \{0\} \rightarrow \mathbb{Z}_{\geq 0}$, there exists a monomial ideal Q for which $\text{depth } S/Q^k = f(k)$ for all $k \geq 1$. Unlike the proof of [3, Theorem 4.1], since the proof of [2, Theorem 4.9] relies on induction on $\lim_{k \rightarrow \infty} f(k)$, no explicit description of a system of generators of a monomial ideal Q is provided.

Our original motivation to organize this paper was to find an explicit description of a system of generators of a monomial ideal Q of [2, Theorem 4.9]. However, there seems to be a gap in the proof of [2, Theorem 4.9] and it is unclear whether

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[2, Theorem 4.9] is true. In fact, the inductive argument done in the proof of [2, Theorem 4.9] cannot be valid for the nonincreasing function $f : \mathbb{Z}_{\geq 0} \setminus \{0\} \rightarrow \mathbb{Z}_{\geq 0}$ with $f(1) = f(2) = 2$ and $f(3) = f(4) = \cdots = 0$. In the present paper, given a nonincreasing function $f : \mathbb{Z}_{\geq 0} \setminus \{0\} \rightarrow \mathbb{Z}_{\geq 0}$ such that

- $f(k) - f(k+1) \leq 1$ for all $k \geq 1$;
- if $a = f(1)$ and $b = \lim_{k \rightarrow \infty} f(k)$, then

$$|f^{-1}(a)| \leq |f^{-1}(a-1)| \leq \cdots \leq |f^{-1}(b+1)|,$$

a system of generators of a monomial ideal I for which $\text{depth } S/I^k = f(k)$ for all $k \geq 1$ is explicitly described (Theorem 1.1). Furthermore, we give a characterization of triplets of integers (n, d, r) with $n > 0$, $d \geq 0$ and $r > 0$ with the properties that there exists a monomial ideal $I \subset S = K[x_1, \dots, x_n]$ for which $\lim_{k \rightarrow \infty} \text{depth } S/I^k = d$ and $\text{dstab}(I) = r$ (Theorem 2.1).

1. NONINCREASING DEPTH FUNCTIONS

Let K be a field and $S = K[x_1, \dots, x_n]$ the polynomial ring in n variables over K with each $\deg x_i = 1$.

In this section, we show the following theorem.

Theorem 1.1. *Given a nonincreasing function $f : \mathbb{Z}_{\geq 0} \setminus \{0\} \rightarrow \mathbb{Z}_{\geq 0}$ such that*

- $f(k) - f(k+1) \leq 1$ for all $k \geq 1$;
- if $a = f(1)$ and $b = \lim_{k \rightarrow \infty} f(k)$, then

$$|f^{-1}(a)| \leq |f^{-1}(a-1)| \leq \cdots \leq |f^{-1}(b+1)|,$$

there is a monomial ideal I for which $\text{depth } S/I^k = f(k)$ for all $k \geq 1$.

At first, we prepare some lemmas to prove Theorem 1.1.

Lemma 1.2. ([5, Corollary 5.11]) *Let I be a monomial ideal in S . Then for any integer $k \geq 1$, we have*

$$\text{depth } I^{k-1}/I^k = \min\{\text{depth } I^{k-1}, \text{depth } I^k - 1\}.$$

Lemma 1.3. *Let I be a monomial ideal in S . Then the following arguments are equivalent:*

- (a) $\text{depth } S/I^k$ is nonincreasing.
- (b) $\text{depth } I^{k-1}/I^k$ is nonincreasing.

Moreover, when this is the case, $\text{depth } S/I^k = \text{depth } I^{k-1}/I^k$ for any $k \geq 1$.

Proof. Set $f(k) = \text{depth } S/I^k$ and $g(k) = \text{depth } I^{k-1}/I^k$. Since we obtain $\text{depth } I^k = \text{depth } S/I^k + 1$ for any $k \geq 1$, by Lemma 1.2, it is obvious that

$$g(k) = \min\{f(k-1) + 1, f(k)\}, k = 1, 2, \dots$$

Hence we know that if $f(k)$ is nonincreasing, then we have $g(k) = f(k)$.

On the other hand, we assume that $g(k)$ is nonincreasing. If $f(t) = g(t)$ for an integer $t \geq 1$, then we have $f(t+1) = g(t+1)$. Since $f(1) = g(1)$, it follows that for any integer $k \geq 1$, $f(k) = g(k)$. \square

Lemma 1.4. *Set $A = K[x_1, \dots, x_{n'}]$ and $B = K[x_{n'+1}, \dots, x_n]$, and we let I, J are monomial ideals in A and B . Then for any integer $t \geq 1$, we have*

$$\text{depth}(I+J)^{t-1}/(I+J)^t = \min_{\substack{i+j=t+1 \\ i,j \geq 1}} \{\text{depth } I^{i-1}/I^i + \text{depth } J^{j-1}/J^j\}.$$

Proof. It follows by combining [2, Theorem3.3 (i)] and [5, Theorem1.1]. \square

The following proposition is important in this paper.

Proposition 1.5. *Let $t \geq 2$ be an integer and we set a monomial ideal $I = (x^t, xy^{t-2}z, y^{t-1}z)$ in $B = K[x, y, z]$. Then*

$$\text{depth } B/I^n = \begin{cases} 1, & \text{if } n \leq t-1, \\ 0, & \text{if } n \geq t. \end{cases}$$

Proof. First of all, for each integer $n \geq t$, we show that $\text{depth } B/I^n = 0$. For this purpose we find a monomial belonging to $(I^n : \mathfrak{m}) \setminus I^n$, where $\mathfrak{m} = (x, y, z)$. We claim that the monomial $u = x^{tn-t^2+t}y^{t^2-2t}z^{t-1}$ belongs to $(I^n : \mathfrak{m}) \setminus I^n$. Indeed, each generator of I^n forms

$$w(a, b, c) := (x^t)^a(xy^{t-2}z)^b(y^{t-1}z)^c = x^{ta+b}y^{(t-2)b+(t-1)c}z^{b+c},$$

where $a + b + c = n$ and $a, b, c \geq 0$. Then we have

$$w(n-t+1, 1, t-2)|xu,$$

$$w(n-t+1, 0, t-1)|yu,$$

$$w(n-t, t, 0)|zu.$$

Thus $u \in (I^n : \mathfrak{m})$. While the degree of u is less than that of generators in I^n . Hence we obtain $u \notin I^n$.

Next, we show that $\text{pd } I^n = 1$ for all $1 \leq n \leq t-1$. In order to prove this, we use the theory of *Buchberger graphs*. Let m_1, \dots, m_s be the generators of I^n . The Buchberger graph $\text{Buch}(I^n)$ has vertices $1, \dots, s$ and an edge (i, j) whenever there is no monomial m_k such that m_k divides $\text{lcm}(m_i, m_j)$ and the degree of m_k is different from $\text{lcm}(m_i, m_j)$ in every variable that occurs in $\text{lcm}(m_i, m_j)$. Then it is known that the syzygy module $\text{syz}(I^n)$ is generated by syzygies

$$\sigma_{ij} = \frac{\text{lcm}(m_i, m_j)}{m_i} \mathbf{e}_i - \frac{\text{lcm}(m_i, m_j)}{m_j} \mathbf{e}_j$$

corresponding to edges (i, j) in $\text{Buch}(I^n)$ ([4, Proposition 3.5]).

Let $G(I^n) := \{w(a, b, c) = x^{ta+b}y^{(t-2)b+(t-1)c}z^{b+c} \mid a, b, c \geq 0, a+b+c = n\}$ be the set of generators of I^n . We introduce the following lexicographic order $<$ on $G(I^n)$. Let $w(a, b, c), w(a', b', c') \in G(I^n)$. Then we define

- $w(a', b', c') < w(a, b, c)$ if $a' < a$;
- $w(a', b', c') < w(a, b, c)$ if $a' = a$ and $b' < b$.

Observation 1.6. For $w = x^a y^b z^c$, we denote $\deg_x w = a$, $\deg_y w = b$ and $\deg_z w = c$. It is easy to see that

- $\deg_x w(a', b', c') < \deg_x w(a, b, c)$ if and only if $w(a', b', c') < w(a, b, c)$;
- $\deg_y w(a', b', c') \geq \deg_y w(a, b, c)$ if $w(a', b', c') < w(a, b, c)$;
- $\deg_z w(a', b', c') \geq \deg_z w(a, b, c)$ if $w(a', b', c') < w(a, b, c)$

if $1 \leq n \leq t-1$.

To construct the minimal free resolution of I^n , we compute generators of $\text{syz}(I^n)$. For $w(a, b, c), w(a', b', c') \in G(I^n)$, we define $w(a', b', c') \leq w(a, b, c)$ if $w(a', b', c') < w(a, b, c)$ and there is no monomial $w \in G(I^n)$ such that $w(a', b', c') < w < w(a, b, c)$. Moreover, we put

$$\begin{aligned} & \sigma((a, b, c), (a', b', c')) \\ := & \frac{\text{lcm}(w(a, b, c), w(a', b', c'))}{w(a, b, c)} \mathbf{e}_{(a, b, c)} - \frac{\text{lcm}(w(a, b, c), w(a', b', c'))}{w(a', b', c')} \mathbf{e}_{(a', b', c')}. \end{aligned}$$

We show that

Claim 1. If $w(a', b', c') \leq w(a, b, c)$, then $\{w(a', b', c'), w(a, b, c)\}$ is an edge of $\text{Buch}(I^n)$.

Proof of Claim 1. Note that $w(a', b', c') \leq w(a, b, c)$ if and only if either $a' = a, b' = b-1$ and $c' = c+1$ or $(a, b, c) = (a, 0, n-a)$ and $(a', b', c') = (a-1, n-a+1, 0)$. In the former case, we have $\text{lcm}(w(a, b, c), w(a, b-1, c+1)) = x^{ta+b} y^{(t-2)(b-1)+(t-1)(c+1)} z^{n-a}$ from Observation 1.6. It is enough to show that there is no monomial $w \in G(I^n)$ such that $w \mid \text{lcm}(w(a, b, c), w(a, b-1, c+1))/xyz = x^{ta+b-1} y^{(t-2)(b-1)+(t-1)(c+1)-1} z^{n-a-1}$.

Assume that there exists such a monomial $w \in G(I^n)$. Then $\deg_x w \leq ta + b - 1$. Hence $w \leq w(a, b-1, c+1)$ from Observation 1.6. However, $\deg_z w \geq b + c = n - a$ from Observation 1.6 again, this is a contradiction.

Next, we consider the latter case, that is, $(a, b, c) = (a, 0, n-a)$ and $(a', b', c') = (a-1, n-a+1, 0)$. As in the former case, it is enough to show that there is no monomial $w \in G(I^n)$ such that $w \mid \text{lcm}(w(a, 0, n-a), w(a-1, n-a+1, 0))/xyz = x^{ta-1} y^{(t-2)(n-a+1)-1} z^{n-a}$. Assume that there exists such a monomial $w \in G(I^n)$. Then $\deg_x w \leq ta - 1$ and $w \leq w(a-1, n-a+1, 0)$ from Observation 1.6. But we have $\deg_z w \geq n - a + 1$ from Observation 1.6 again, this is a contradiction.

Therefore, we have the desired conclusion. \square

Here, we put $\Sigma := \{\sigma((a, b, c), (a', b', c')) \mid w(a', b', c') \leq w(a, b, c)\}$. Next, we will show the following:

Claim 2. Assume that $w(a', b', c') < w(a, b, c)$ and $w(a', b', c') \not\leq w(a, b, c)$. Then $\sigma((a, b, c), (a', b', c'))$ can be expressed as an S -linear combination of the elements of Σ .

Proof of Claim 2. Let $s \geq 3$ and assume that

$$w(a', b', c') = w(a_s, b_s, c_s) \leq w(a_{s-1}, b_{s-1}, c_{s-1}) \leq \cdots \leq w(a_1, b_1, c_1) = w(a, b, c).$$

From Observation 1.6, we can see that

$$\frac{\text{lcm}(w(a_1, b_1, c_1), w(a_s, b_s, c_s))}{\text{lcm}(w(a_i, b_i, c_i), w(a_{i+1}, b_{i+1}, c_{i+1}))}$$

is a monomial in S for all $1 \leq i \leq s-1$. Hence we have

$$\begin{aligned} \sigma((a, b, c), (a', b', c')) &= \sigma((a_1, b_1, c_1), (a_s, b_s, c_s)) \\ &= \sum_{i=1}^{s-1} \frac{\text{lcm}(w(a_1, b_1, c_1), w(a_s, b_s, c_s))}{\text{lcm}(w(a_i, b_i, c_i), w(a_{i+1}, b_{i+1}, c_{i+1}))} \sigma((a_i, b_i, c_i), (a_{i+1}, b_{i+1}, c_{i+1})). \end{aligned}$$

Thus we have the desired conclusion. \square

By Claim 1, 2 and [4, Proposition 3.5], Σ is the set of generators of $\text{syz}(I^n)$. Moreover, it is clear that the elements of Σ are linearly independent on S . Hence

$$0 \rightarrow \bigoplus_j S(-j)^{\beta_{1,j}} \rightarrow S(-nt)^{\beta_{0,nt}} \rightarrow I^n \rightarrow 0$$

is the minimal free resolution of I^n . Therefore we have $\text{pd } I^n = 1$. \square

Now, we can prove Theorem 1.1.

Proof of Theorem 1.1. First, for any integers $i, k \geq 1$, we define the monomial ideal $I_{k,i} := (x_i^{k+1}, x_i y_i^{k-1} z_i, y_i^k z_i)$ in $B_i = K[x_i, y_i, z_i]$. Then by Proposition 1.5, we obtain

$$\text{depth } B_i / I_{k,i}^t = \begin{cases} 1, & \text{if } t \leq k, \\ 0 & \text{if } t > k. \end{cases}$$

Set $n = a - b$ and $s_i := |f^{-1}(a - i + 1)|$ for each $1 \leq i \leq n$. We show that $I = \sum_{i=1}^n I_{s_i, i}$ in $S = K[x_1, y_1, z_1, \dots, x_n, y_n, z_n, w_1, \dots, w_b]$ is the required monomial ideal. By Lemma 1.3 and 1.4, we immediately show the assertion follows. \square

Example 1.7. Nonincreasing functions $f : \mathbb{Z}_{\geq 0} \setminus \{0\} \rightarrow \mathbb{Z}_{\geq 0}$ with $f(1) = f(2) = 2$ and $f(3) = f(4) = \cdots = 0$ and $g : \mathbb{Z}_{\geq 0} \setminus \{0\} \rightarrow \mathbb{Z}_{\geq 0}$ with $g(1) = g(2) = 2$, $g(3) = 1$ and $g(4) = g(5) = \cdots = 0$ do not satisfy the assumption of Theorem 1.1. However there exist monomial ideals I, J of $S = K[x_1, \dots, x_6]$ such that $\text{depth } S/I^k = f(k)$ and $\text{depth } S/J^k = g(k)$ for $k \geq 1$.

Indeed, $I = (x_1^3, x_1x_2x_3, x_2^2x_3)(x_4^3, x_4x_5x_6, x_5^2x_6) + (x_1^4, x_1^3x_2, x_1x_2^3, x_2^4, x_1^2x_2^2x_3)$ and $J = (x_1^4, x_1x_2^2x_3, x_2^3x_3)(x_4^4, x_4x_5^2x_6, x_5^3x_6) + (x_1^5, x_1^4x_2, x_1x_2^4, x_2^5, x_1^3x_2^2x_3)$ are the desired monomial ideals.

2. THE NUMBER OF VARIABLES AND DEPTH STABILITY NUMBER

Let $I \neq (0)$ be a monomial ideal in $S = K[x_1, \dots, x_n]$ and $f(k)$ the depth function of I . We set $\lim_{k \rightarrow \infty} f(k) = d$ and $r = \text{dstab}(I)$. When $n = 1$, we know that $d = 0$ and $r = 1$. Moreover, when $n = 2$, we have $0 \leq d \leq 1$ and $r = 1$.

In this section, for $n \geq 3$, we discuss bounds of the limit depth and depth stability number of a monomial ideal. In fact, we show the following theorem.

Theorem 2.1. *Assume $n \geq 3$. Let $I \neq (0)$ be a monomial ideal in $S = K[x_1, \dots, x_n]$ and $f(k)$ the depth function of I . We set $\lim_{k \rightarrow \infty} f(k) = d$ and $r = \text{dstab}(I)$. Then one of the followings is satisfied:*

- $0 \leq d \leq n - 2$ and $r \geq 1$.
- $d = n - 1$ and $r = 1$.

Conversely, for any d and r satisfied one of the above, there exists a monomial ideal J in S such that $\lim_{k \rightarrow \infty} g(k) = d$ and $r = \text{dstab}(J)$, where $g(k)$ is the depth function of J .

Proof. In general, for any monomial ideal $I \neq (0)$ in S , we have $0 \leq \text{depth } S/I \leq n - 1$. We assume that $d = n - 1$. Since $\dim S/I^r \leq n - 1$, S/I^r is Cohen-Macaulay. Hence for any minimal prime ideal P of I^r , we have $\text{height } P = 1$. In particular, P is a principle ideal since S is UFD. Hence I^r is a principle ideal. This says that I is also a principle ideal. Thus, for any $k \geq 1$, S/I^k is a hypersurface. Therefore, we have $r = 1$.

Next, we show the latter part. Assume that $0 \leq d \leq n - 3$ and $r \geq 2$. Let $J_1 = (x_1^r, x_1x_2^{r-2}x_3, x_2^{r-1}x_3) \subset A := K[x_1, x_2, x_3]$. By Proposition 1.5, we have

$$\text{depth } A/J_1^k = \begin{cases} 0, & \text{if } k \geq r, \\ 1, & \text{if } k \leq r - 1. \end{cases}$$

Let $J = J_1 + (x_4, \dots, x_{n-d}) = (x_1^r, x_1x_2^{r-2}x_3, x_2^{r-1}x_3, x_4, \dots, x_{n-d})$ be a monomial ideal in S and $g_1(k)$ the depth function of J . Then we have $\lim_{k \rightarrow \infty} g_1(k) = d$ and $\text{dstab}(J) = r$. Moreover, an ideal $J_2 = (x_1, \dots, x_{n-d}) \subset S$ satisfies that $\text{depth}(S/J_2^k) = d$ for all $k \geq 1$, that is, $\lim_{k \rightarrow \infty} \text{depth}(S/J_2^k) = d$ and $\text{dstab}(J_2) = 1$.

Next, we assume that $d = n - 2$ and $r \geq 1$. By [3, Proof of Theorem 4.1], we can see that a monomial ideal $J_3 = (x_1^{r+2}, x_1^{r+1}x_2, x_1x_2^{r+1}, x_2^{r+2}, x_1^rx_2^2x_3) \subset A$ satisfies that $\text{dstab}(J_3) = r$ and

$$\text{depth } A/J_3^k = \begin{cases} 1, & \text{if } k \geq r, \\ 0, & \text{if } k \leq r - 1. \end{cases}$$

Let $J' = J_3$ be the monomial ideal in S and $g_2(k)$ the depth function of J' . Then we have $\lim_{k \rightarrow \infty} g_2(k) = d$ and $\text{dstab}(J') = r$.

When $d = n - 1$ and $r = 1$, we immediately obtain a monomial ideal satisfied the condition by the former part of this proof, as desired. \square

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